BERNSTEIN-SZEGÖ POLYNOMIALS ASSOCIATED WITH ROOT SYSTEMS

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Abstract. We introduce multivariate generalizations of the Bernstein-Szegö polynomials, which are associated to the root systems of the complex simple Lie algebras. The multivariate polynomials in question generalize Macdonald's Hall-Littlewood polynomials associated with root systems. For the root system of type A_1 (corresponding to the Lie algebra $\mathfrak{sl}(2;\mathbb{C})$) the classic Bernstein-Szegö polynomials are recovered.

1. Introduction

Recent years have revealed the birth of an elegant multivariate generalization of the classical theory of (basic) hypergeometric orthogonal polynomials based on the root systems of complex simple Lie algebras [O, HS, M3, DX, M4]. The purpose of the present paper is to introduce—in the same spirit—a multivariate generalization of the Bernstein-Szegö polynomials [S].

By definition, Bernstein-Szegö polynomials $p_\ell(x)$, $\ell = 0, 1, 2, \ldots$, are the trigonometric orthogonal polynomials obtained by Gram-Schmidt orthogonalization of the Fourier-cosine basis $m_\ell(x) = \exp(i\ell x) + \exp(-i\ell x)$, $\ell = 0, 1, 2, \ldots$, with respect to the inner product

$$\langle m_\ell, m_k\rangle_\Delta = \frac{1}{2\pi}\int_0^\pi m_\ell(x)\overline{m_k(x)}\Delta(x)\mathrm{d}x, \tag{1.1a}$$

characterized by the nonnegative rational trigonometric weight function of the form

$$\Delta(x) = \frac{|\delta(x)|^2}{c(x)c(-x)}, \quad c(x) = \prod_{m=1}^{M}(1 + t_m e^{-2ix}), \tag{1.1b}$$

where $\delta(x) := \exp(ix) - \exp(-ix)$. Here the parameters $t_1, \ldots, t_M$ are assumed to lie in the domain $(-1, 1) \setminus \{0\}$. A crucial property of the polynomials in question is that—for sufficiently large degree ℓ—they are given explicitly by the compact formula [S]

$$p_\ell(x) = \frac{1}{\mathcal{N}_\ell\delta(x)}\left(c(x)e^{i(\ell+1)x} - c(-x)e^{-i(\ell+1)x}\right), \qquad \ell \geq M-1, \tag{1.2a}$$

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where

$$\mathcal{N}_\ell = \begin{cases} 1 - t_1 \cdots t_M & \text{if } \ell = M-1, \\ 1 & \text{if } \ell \geq M. \end{cases} \tag{1.2b}$$

Furthermore, the quadratic norms of the corresponding Bernstein-Szegö polynomials are given by [S]

$$\langle p_\ell, p_\ell \rangle_\Delta = \mathcal{N}_\ell^{-1}, \qquad \ell \geq M-1. \tag{1.3}$$

The main result of this paper is a multivariate generalization of these formulas associated with the root systems of the complex simple Lie algebras (cf. Theorems 2.1, 2.2 and 2.3 below). The classical formulas in Eqs. (1.2a)–(1.3) are recovered from our results upon specialization to the case of the Lie algebra $\mathfrak{sl}(2;\mathbb{C})$ (corresponding to the root system A_1). Particular instances of the multivariate Bernstein-Szegö polynomials discussed here have previously surfaced in Refs. [R, Di], in the context of a study of the large-degree asymptotics of Macdonald's multivariate basic hypergeometric orthogonal polynomials related to root systems [M3, M4]. The simplest examples of our multivariate Bernstein-Szegö polynomials—corresponding to weight functions characterized by c-functions of degree $M = 0$ and degree $M = 1$, respectively—amount to the celebrated Weyl characters and to Macdonald's Hall-Littlewood polynomials associated with root systems [M1, M3].

The paper is organized as follows. In Section 2 the main results are stated. The remainder of the paper, viz. Sections 3–5, is devoted to the proofs.

Note. Throughout we will make extensive use of the language of root systems. For preliminaries and further background material on root systems the reader is referred to e.g. Refs. [B, Hu].

2. Bernstein-Szegö polynomials for root systems

Let $\mathbf{E}$ be a real finite-dimensional Euclidian vector space with scalar product $\langle \cdot, \cdot \rangle$, and let $\mathbf{R}$ denote an irreducible crystallographic root system spanning $\mathbf{E}$. Throughout it is assumed that $\mathbf{R}$ be *reduced*. We will employ the following standard notational conventions for the dual root system $\mathbf{R}^\vee := \{\boldsymbol{\alpha}^\vee \mid \boldsymbol{\alpha} \in \mathbf{R}\}$ (where $\boldsymbol{\alpha}^\vee := 2\boldsymbol{\alpha}/\langle \boldsymbol{\alpha}, \boldsymbol{\alpha} \rangle$), the root lattice $\mathcal{Q} := \mathrm{Span}_{\mathbb{Z}}(\mathbf{R})$ and its nonnegative semigroup $\mathcal{Q}_+ := \mathrm{Span}_{\mathbb{N}}(\mathbf{R}_+)$ generated by the positive roots $\mathbf{R}_+$; the duals of the latter two objects are given by the weight lattice $\mathcal{P} := \{\boldsymbol{\lambda} \in \mathbf{E} \mid \langle \boldsymbol{\lambda}, \boldsymbol{\alpha}^\vee \rangle \in \mathbb{Z},\ \forall \boldsymbol{\alpha} \in \mathbf{R}\}$ and its dominant integral cone $\mathcal{P}_+ := \{\boldsymbol{\lambda} \in \mathcal{P} \mid \langle \boldsymbol{\lambda}, \boldsymbol{\alpha}^\vee \rangle \in \mathbb{N},\ \forall \boldsymbol{\alpha} \in \mathbf{R}_+\}$. Finally, we denote by W the Weyl group generated by the orthogonal reflections $r_{\boldsymbol{\alpha}} : \mathbf{E} \to \mathbf{E}$, $\boldsymbol{\alpha} \in \mathbf{R}$ in the hyperplanes perpendicular to the roots (so for $\mathbf{x} \in \mathbf{E}$ one has that $r_{\boldsymbol{\alpha}}(\mathbf{x}) = \mathbf{x} - \langle \mathbf{x}, \boldsymbol{\alpha}^\vee \rangle \boldsymbol{\alpha}$). Clearly $\|w\mathbf{x}\|^2 = \langle w\mathbf{x}, w\mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle = \|\mathbf{x}\|^2$ for all $w \in W$ and $\mathbf{x} \in \mathbf{E}$.

The algebra $\boldsymbol{A}_{\mathbf{R}}$ of Weyl-group invariant trigonometric polynomials on the torus $\mathbb{T}_{\mathbf{R}} = \mathbf{E}/(2\pi\mathcal{Q}^\vee)$ (where $\mathcal{Q}^\vee := \mathrm{Span}_{\mathbb{Z}}(\mathbf{R}^\vee)$) is spanned by the basis of the symmetric monomials

$$m_{\boldsymbol{\lambda}}(\mathbf{x}) = \frac{1}{|W_{\boldsymbol{\lambda}}|} \sum_{w \in W} e^{i\langle \boldsymbol{\lambda}, \mathbf{x}_w \rangle}, \qquad \boldsymbol{\lambda} \in \mathcal{P}_+. \tag{2.1}$$

Here $|W_{\boldsymbol{\lambda}}|$ represents the order of stabilizer subgroup $W_{\boldsymbol{\lambda}} := \{w \in W \mid w(\boldsymbol{\lambda}) = \boldsymbol{\lambda}\}$ and $\mathbf{x}_w := w(\mathbf{x})$. We endow $\boldsymbol{A}_{\mathbf{R}}$ with the following inner product structure

$$\langle f, g \rangle_\Delta = \frac{1}{|W| \mathrm{Vol}(\mathbb{T}_{\mathbf{R}})} \int_{\mathbb{T}_{\mathbf{R}}} f(\mathbf{x}) \overline{g(\mathbf{x})} \Delta(\mathbf{x}) \mathrm{d}\mathbf{x} \qquad (f, g \in \boldsymbol{A}_{\mathbf{R}}), \tag{2.2a}$$

associated to a W-invariant nonnegative weight function that factorizes over the root system:

$$\Delta(\mathbf{x}) = \frac{|\delta(\mathbf{x})|^2}{C(\mathbf{x})C(-\mathbf{x})}, \tag{2.2b}$$

$$\delta(\mathbf{x}) = \prod_{\boldsymbol{\alpha}\in\mathbf{R}_+} \left(e^{i\langle\boldsymbol{\alpha},\mathbf{x}\rangle/2} - e^{-i\langle\boldsymbol{\alpha},\mathbf{x}\rangle/2}\right), \tag{2.2c}$$

$$C(\mathbf{x}) = \prod_{\boldsymbol{\alpha}\in\mathbf{R}_+^{(s)}} c^{(s)}(e^{-i\langle\boldsymbol{\alpha},\mathbf{x}\rangle}) \prod_{\boldsymbol{\alpha}\in\mathbf{R}_+^{(l)}} c^{(l)}(e^{-i\langle\boldsymbol{\alpha},\mathbf{x}\rangle}), \tag{2.2d}$$

where

$$c^{(s)}(z) = \prod_{m=1}^{M^{(s)}} (1 + t_m^{(s)} z), \qquad c^{(l)}(z) = \prod_{m=1}^{M^{(l)}} (1 + t_m^{(l)} z), \tag{2.2e}$$

and with the parameters $t_m^{(s)}$ $(m = 1, \ldots, M^{(s)})$ and $t_m^{(l)}$ $(m = 1, \ldots, M^{(l)})$ taken from $(-1, 1) \setminus \{0\}$. Here $|W|$ denotes the order of the Weyl group W, $\mathrm{Vol}(\mathbb{T}_{\mathbf{R}}) := \int_{\mathbb{T}_{\mathbf{R}}} \mathrm{d}\mathbf{x}$, and $\mathbf{R}_+^{(s)} := \mathbf{R}^{(s)} \cap \mathbf{R}_+$, $\mathbf{R}_+^{(l)} := \mathbf{R}^{(l)} \cap \mathbf{R}_+$, where $\mathbf{R}^{(s)}$ and $\mathbf{R}^{(l)}$ refer to the *short roots* and the *long roots* of $\mathbf{R}$, respectively (with the convention that all roots are short, say, if $\mathbf{R}$ is simply-laced).

The Bernstein-Szegö polynomials associated to the root system $\mathbf{R}$ are now defined as the polynomials obtained from the symmetric monomials $m_{\boldsymbol{\lambda}}(\mathbf{x})$, $\boldsymbol{\lambda} \in \mathcal{P}_+$ by projecting away the components in the finite-dimensional subspace spanned by monomials corresponding to dominant weights that are smaller than $\boldsymbol{\lambda}$ in the (partial) *dominance ordering*

$$\boldsymbol{\mu} \preceq \boldsymbol{\lambda} \quad \text{iff} \quad \boldsymbol{\lambda} - \boldsymbol{\mu} \in \mathcal{Q}_+. \tag{2.3}$$

Definition. The *(monic) Bernstein-Szegö polynomials* $p_{\boldsymbol{\lambda}}(\mathbf{x})$, $\boldsymbol{\lambda} \in \mathcal{P}_+$ are the polynomials of the form

$$p_{\boldsymbol{\lambda}}(\mathbf{x}) = \sum_{\boldsymbol{\mu}\in\mathcal{P}^+,\, \boldsymbol{\mu}\preceq\boldsymbol{\lambda}} a_{\boldsymbol{\lambda}\boldsymbol{\mu}} m_{\boldsymbol{\mu}}(\mathbf{x}), \tag{2.4a}$$

with expansion coefficients $a_{\boldsymbol{\lambda}\boldsymbol{\mu}} \in \mathbb{C}$ such that $a_{\boldsymbol{\lambda}\boldsymbol{\lambda}} = 1$ and

$$\langle p_{\boldsymbol{\lambda}}, m_{\boldsymbol{\mu}}\rangle_\Delta = 0 \quad \text{for } \boldsymbol{\mu} \prec \boldsymbol{\lambda}. \tag{2.4b}$$

It is clear that for any $\boldsymbol{\lambda} \in \mathcal{P}_+$ the properties in Eqs. (2.4a), (2.4b) determine $p_{\boldsymbol{\lambda}}(\mathbf{x})$ uniquely. The main result of this paper is an explicit formula for the Bernstein-Szegö polynomials for weights $\boldsymbol{\lambda}$ sufficiently deep in the dominant cone $\mathcal{P}_+$. To formulate the precise result we introduce the quantities

$$m^{(s)}(\boldsymbol{\lambda}) = \min_{\boldsymbol{\alpha}\in\mathbf{R}_+^{(s)}} \langle\boldsymbol{\lambda}, \boldsymbol{\alpha}^\vee\rangle \quad \text{and} \quad m^{(l)}(\boldsymbol{\lambda}) = \min_{\boldsymbol{\alpha}\in\mathbf{R}_+^{(l)}} \langle\boldsymbol{\lambda}, \boldsymbol{\alpha}^\vee\rangle, \tag{2.5}$$

which measure the distance of the dominant weight $\boldsymbol{\lambda}$ to the walls $\{\boldsymbol{\mu} \in \mathcal{P}_+ \mid \exists \boldsymbol{\alpha} \in \mathbf{R}_+ \text{ such that } \langle\boldsymbol{\mu}, \boldsymbol{\alpha}^\vee\rangle = 0\}$ bounding the dominant cone. For future reference we will also single out the special dominant weights given by the half-sums of the positive roots:

$$\boldsymbol{\rho} := \frac{1}{2}\sum_{\boldsymbol{\alpha}\in\mathbf{R}_+} \boldsymbol{\alpha}, \qquad \boldsymbol{\rho}^{(s)} := \frac{1}{2}\sum_{\boldsymbol{\alpha}\in\mathbf{R}_+^{(s)}} \boldsymbol{\alpha}, \qquad \boldsymbol{\rho}^{(l)} := \frac{1}{2}\sum_{\boldsymbol{\alpha}\in\mathbf{R}_+^{(l)}} \boldsymbol{\alpha}. \tag{2.6}$$

Definition. Let us call a weight $\boldsymbol{\lambda} \in \mathcal{P}_+$ *sufficiently deep* in the dominant cone iff

$$m^{(s)}(\boldsymbol{\lambda}) \geq M^{(s)} - 1 \quad \text{and} \quad m^{(l)}(\boldsymbol{\lambda}) \geq M^{(l)} - 1 \tag{2.7}$$

(where $M^{(s)}$ and $M^{(l)}$ refer to the degrees of $c^{(s)}(z)$ and $c^{(l)}(z)$ in Eq. (2.2e), respectively).

Theorem 2.1 (Explicit Formula)**.** *For $\boldsymbol{\lambda} \in \mathcal{P}_+$ sufficiently deep in the dominant cone, the monic Bernstein-Szegö polynomial $p_{\boldsymbol{\lambda}}(\mathbf{x})$* (2.4a), (2.4b) *is given explicitly by*

$$p_{\boldsymbol{\lambda}}(\mathbf{x}) = \mathcal{N}_{\boldsymbol{\lambda}}^{-1} P_{\boldsymbol{\lambda}}(\mathbf{x}) \quad \textit{with} \quad P_{\boldsymbol{\lambda}}(\mathbf{x}) = \frac{1}{\delta(\mathbf{x})} \sum_{w \in W} (-1)^w C(\mathbf{x}_w) e^{i\langle \boldsymbol{\rho} + \boldsymbol{\lambda}, \mathbf{x}_w \rangle}, \tag{2.8a}$$

where $C(\mathbf{x})$ is taken from Eqs. (2.2d),(2.2e) *and $(-1)^w := \det(w)$. Here the normalization constant is of the form*

$$\mathcal{N}_{\boldsymbol{\lambda}} = \prod_{\substack{\boldsymbol{\alpha} \in \mathbf{R}_+^{(s)} \\ \langle \tilde{\boldsymbol{\lambda}}, \boldsymbol{\alpha}^\vee \rangle = 0}} \frac{1 - \mathbf{t}_s^{1 + ht_s(\boldsymbol{\alpha})} \mathbf{t}_l^{ht_l(\boldsymbol{\alpha})}}{1 - \mathbf{t}_s^{ht_s(\boldsymbol{\alpha})} \mathbf{t}_l^{ht_l(\boldsymbol{\alpha})}} \prod_{\substack{\boldsymbol{\alpha} \in \mathbf{R}_+^{(l)} \\ \langle \tilde{\boldsymbol{\lambda}}, \boldsymbol{\alpha}^\vee \rangle = 0}} \frac{1 - \mathbf{t}_s^{ht_s(\boldsymbol{\alpha})} \mathbf{t}_l^{1 + ht_l(\boldsymbol{\alpha})}}{1 - \mathbf{t}_s^{ht_s(\boldsymbol{\alpha})} \mathbf{t}_l^{ht_l(\boldsymbol{\alpha})}}, \tag{2.8b}$$

where $\tilde{\boldsymbol{\lambda}} := \boldsymbol{\lambda} + \boldsymbol{\rho} - M^{(s)} \boldsymbol{\rho}^{(s)} - M^{(l)} \boldsymbol{\rho}^{(l)}$, $\mathbf{t}_s := -t_1^{(s)} \cdots t_{M^{(s)}}^{(s)}$, $\mathbf{t}_l := -t_1^{(l)} \cdots t_{M^{(l)}}^{(l)}$, $ht_s(\boldsymbol{\alpha}) := \sum_{\boldsymbol{\beta} \in \mathbf{R}_+^{(s)}} \langle \boldsymbol{\alpha}, \boldsymbol{\beta}^\vee \rangle / 2$ and $ht_l(\boldsymbol{\alpha}) := \sum_{\boldsymbol{\beta} \in \mathbf{R}_+^{(l)}} \langle \boldsymbol{\alpha}, \boldsymbol{\beta}^\vee \rangle / 2$ (and with the convention that empty products are equal to one*).*

It is immediate from the definition that the Bernstein-Szegö polynomials are orthogonal when corresponding to weights that are comparable in the dominance ordering (2.3). The following theorem states that the orthogonality holds in fact also for non-comparable weights, assuming at least one of them lies sufficiently deep in the dominant cone.

Theorem 2.2 (Orthogonality)**.** *When at least one of $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_+$ lies sufficiently deep in the dominant cone, the Bernstein-Szegö polynomials* (2.4a),(2.4b) *are orthogonal*

$$\langle p_{\boldsymbol{\lambda}}, p_{\boldsymbol{\mu}} \rangle_\Delta = 0 \quad \textit{if} \quad \boldsymbol{\mu} \neq \boldsymbol{\lambda}. \tag{2.9}$$

Our final result provides an explicit formula for the quadratic norm of the Bernstein-Szegö polynomials corresponding to weights sufficiently deep in the dominant cone.

Theorem 2.3 (Norm Formula)**.** *For $\boldsymbol{\lambda} \in \mathcal{P}_+$ sufficiently deep in the dominant cone, the quadratic norm of the monic Bernstein-Szegö polynomial* (2.4a),(2.4b) *is given by*

$$\langle p_{\boldsymbol{\lambda}}, p_{\boldsymbol{\lambda}} \rangle_\Delta = \mathcal{N}_{\boldsymbol{\lambda}}^{-1} \tag{2.10}$$

(with $\mathcal{N}_{\boldsymbol{\lambda}}$ given by Eq. (2.8b)*).*

For $M^{(s)} = M^{(l)} = 0$ the above Bernstein-Szegö polynomials boil down to the Weyl characters $\chi_{\boldsymbol{\lambda}}(\mathbf{x})$, $\boldsymbol{\lambda} \in \mathcal{P}_+$ of the irreducible representations of simple Lie algebras; and for $M^{(s)} = M^{(l)} = 1$ they amount to Macdonald's Hall-Littlewood polynomials associated with root systems. In these two simplest cases the contents of Theorems 2.1, 2.2 and 2.3 is well-known from the representation theory of simple Lie algebras [Hu] and from Macdonald's seminal work on the zonal spherical functions on p-adic symmetric spaces [M1, M3], respectively. Notice in this connection

that in these two special cases *all* dominant weights are automatically sufficiently deep.

Remark. i. The weights $\boldsymbol{\lambda}$ sufficiently deep in the dominant cone amount precisely to the *dominant* weights of the form $\boldsymbol{\lambda} = \tilde{\boldsymbol{\lambda}} + (M^{(s)} - 1)\boldsymbol{\rho}^{(s)} + (M^{(l)} - 1)\boldsymbol{\rho}^{(l)}$ with $\tilde{\boldsymbol{\lambda}} \in \mathcal{P}_+$.

Remark. ii. When the dominant weights $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ are not comparable in the dominance ordering $\preceq$ (2.3) and moreover neither lies sufficiently deep in the dominant cone, then there is no a priori reason for the orthogonality in Eq. (2.9) to hold. Indeed, computer experiments for small rank indicate that orthogonality may indeed be violated in this situation. However, if one would replace in the definition of the Bernstein-Szegö polynomials given by Eqs. (2.4a), (2.4b) the dominance ordering by a *linear ordering* that is compatible (i.e. extends) $\preceq$ (2.3), then one would end up with an orthogonal basis that coincides with our basis of Bernstein-Szegö polynomials for weights $\boldsymbol{\lambda}$ sufficiently deep. Clearly such a construction would depend (where the weight is not sufficiently deep) on the choice of the linear extension of the dominance ordering $\preceq$ (2.3).

Remark. iii. The classical one-variable Bernstein-Szegö polynomials play an important role in the study of the large-degree asymptotics of orthogonal polynomials on the unit circle [S]. In a nutshell, the idea is that the weight function of the family of orthogonal polynomials whose asymptotics one would like to determine can be approximated (assuming certain analyticity conditions) by the weight function $\Delta(x)$ (1.1b) for $M \to +\infty$ and a suitable choice of the t-parameters. The explicit formula for the Bernstein-Szegö polynomials in Eqs. (1.2a), (1.2b) then converges to the asymptotic formula for the orthogonal polynomials in question. In [R, Di], special cases of the multivariate Bernstein-Szegö polynomials studied in the present paper were employed to compute—in an analogous manner—the asymptotics of families of multivariate orthogonal polynomials (associated with root systems) for dominant weights $\boldsymbol{\lambda}$ deep in the Weyl chamber (i.e. with the distance to the walls going to $+\infty$). An important class of multivariate polynomials whose large-degree asymptotics could be determined by means of this method is given by the Macdonald polynomials [M3, M4].

3. Triangularity and Orthogonality

Following the spirit of Macdonald's concise approach towards the Hall-Littlewood polynomials associated with root systems in Ref. [M3, §10], the idea of the proof of Theorem 2.1 is to demonstrate that the explicit formula stated in the theorem satisfies the two properties characterizing the Bernstein-Szegö polynomials given by Eqs. (2.4a), (2.4b). The orthogonality (Theorem 2.2) and the norm formulas (Theorem 2.3) are then seen to follow from this explicit formula.

First we verify the triangularity of $P_{\boldsymbol{\lambda}}(\mathbf{x})$ (2.8a) with respect to the monomial basis expressed in Eq. (2.4a).

Proposition 3.1 (Triangularity)**.** *For $\boldsymbol{\lambda} \in \mathcal{P}_+$ sufficiently deep, the expansion of the polynomial $P_{\boldsymbol{\lambda}}(\mathbf{x})$ (2.8a) on the monomial basis is triangular:*

$$P_{\boldsymbol{\lambda}}(\mathbf{x}) = \sum_{\boldsymbol{\mu} \in \mathcal{P}^+,\, \boldsymbol{\mu} \preceq \boldsymbol{\lambda}} c_{\boldsymbol{\lambda}\boldsymbol{\mu}} m_{\boldsymbol{\mu}}(\mathbf{x}),$$

with $c_{\boldsymbol{\lambda}\boldsymbol{\mu}} \in \mathbb{C}$.

Proof. Upon expanding the products in $C(\mathbf{x})$ (2.2d), (2.2e) it becomes evident that $P_{\boldsymbol{\lambda}}(\mathbf{x})$ (2.8a) is built of a linear combination of symmetric functions of the form

$$\delta^{-1}(\mathbf{x}) \sum_{w\in W} (-1)^w e^{i\langle \boldsymbol{\rho}+\boldsymbol{\lambda}-\sum_{\boldsymbol{\alpha}\in\mathbf{R}^+} n_{\boldsymbol{\alpha}}\boldsymbol{\alpha}, \mathbf{x}_w\rangle}, \tag{3.1}$$

with $0 \leq n_{\boldsymbol{\alpha}} \leq M^{(s)}$ for $\boldsymbol{\alpha} \in \mathbf{R}^{(s)}_+$ and $0 \leq n_{\boldsymbol{\alpha}} \leq M^{(l)}$ for $\boldsymbol{\alpha} \in \mathbf{R}^{(l)}_+$. The expression in Eq. (3.1) vanishes if $\boldsymbol{\rho}+\boldsymbol{\lambda}-\sum_{\boldsymbol{\alpha}\in\mathbf{R}^+} n_{\boldsymbol{\alpha}}\boldsymbol{\alpha}$ is a singular weight and it is equal—possibly up to a sign—to a Weyl character $\chi_{\boldsymbol{\mu}}(\mathbf{x}) := \delta^{-1}(\mathbf{x})\sum_{w\in W}(-1)^w e^{\langle \boldsymbol{\rho}+\boldsymbol{\mu}, \mathbf{x}_w\rangle}$ otherwise, where $\boldsymbol{\mu}$ denotes the unique dominant weight in the translated Weyl orbit $W(\boldsymbol{\rho}+\boldsymbol{\lambda}-\sum_{\boldsymbol{\alpha}\in\mathbf{R}^+} n_{\boldsymbol{\alpha}}\boldsymbol{\alpha})-\boldsymbol{\rho}$. Since $0 \leq n_{\boldsymbol{\alpha}} \leq M^{(s)} \leq m^{(s)}(\boldsymbol{\lambda})+1 = m^{(s)}(\boldsymbol{\rho}+\boldsymbol{\lambda})$ for $\boldsymbol{\alpha} \in \mathbf{R}^{(s)}_+$ and $0 \leq n_{\boldsymbol{\alpha}} \leq M^{(l)} \leq m^{(l)}(\boldsymbol{\lambda})+1 = m^{(l)}(\boldsymbol{\rho}+\boldsymbol{\lambda})$ for $\boldsymbol{\alpha} \in \mathbf{R}^{(l)}_+$, we can conclude from Proposition 5.2 (below)—upon replacing $\boldsymbol{\lambda}$ by $\boldsymbol{\lambda}+\boldsymbol{\rho}$—that $\boldsymbol{\rho}+\boldsymbol{\lambda}-\sum_{\boldsymbol{\alpha}\in\mathbf{R}^+} n_{\boldsymbol{\alpha}}\boldsymbol{\alpha} \in \mathcal{P}_+(\boldsymbol{\rho}+\boldsymbol{\lambda})$, whence $\boldsymbol{\mu} \preceq \boldsymbol{\lambda}$. This shows that $P_{\boldsymbol{\lambda}}(\mathbf{x})$ (2.8a) is a linear combination of Weyl characters $\chi_{\boldsymbol{\mu}}(\mathbf{x})$ with $\boldsymbol{\mu} \preceq \boldsymbol{\lambda}$. The statement of the proposition is thus clear by the standard fact that Weyl characters expand triangularly on the basis of monomial symmetric functions. □

The next proposition checks (in particular) that $P_{\boldsymbol{\lambda}}(\mathbf{x})$ (2.8a) satisfies the orthogonality relations in Eq. (2.4b).

Proposition 3.2 (Partial Biorthogonality)**.** *For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_+$ such that $\boldsymbol{\mu} \not\succ \boldsymbol{\lambda}$ the polynomial $P_{\boldsymbol{\lambda}}(\mathbf{x})$ (2.8a) and the monomial symmetric function $m_{\boldsymbol{\mu}}(\mathbf{x})$ satisfy the orthogonality relations*

$$\langle P_{\boldsymbol{\lambda}}, m_{\boldsymbol{\mu}}\rangle_\Delta = \begin{cases} 0 & \text{if } \boldsymbol{\mu} \not\succeq \boldsymbol{\lambda}, \\ 1 & \text{if } \boldsymbol{\mu} = \boldsymbol{\lambda}. \end{cases}$$

Proof. An explicit computation starting from the definitions entails that

$$\begin{aligned}
&\langle P_{\boldsymbol{\lambda}}, m_{\boldsymbol{\mu}}\rangle_\Delta = \frac{1}{|W|\,\mathrm{Vol}(\mathbb{T}_{\mathbf{R}})\,|W_{\boldsymbol{\mu}}|} \times \\
&\int_{\mathbb{T}_{\mathbf{R}}} \frac{\delta(-\mathbf{x})}{\mathcal{C}(\mathbf{x})\mathcal{C}(-\mathbf{x})} \sum_{w_1\in W} (-1)^{w_1} \mathcal{C}(\mathbf{x}_{w_1}) e^{i\langle \boldsymbol{\rho}+\boldsymbol{\lambda}, \mathbf{x}_{w_1}\rangle} \sum_{w_2\in W} e^{-i\langle \boldsymbol{\mu}, \mathbf{x}_{w_2}\rangle} \mathrm{d}\mathbf{x} \\
= \; & \frac{1}{\mathrm{Vol}(\mathbb{T}_{\mathbf{R}})\,|W_{\boldsymbol{\mu}}|} \sum_{w\in W} \int_{\mathbb{T}_{\mathbf{R}}} \frac{1}{\mathcal{C}(-\mathbf{x})} \prod_{\boldsymbol{\alpha}\in\mathbf{R}_+} (1-e^{i\langle \boldsymbol{\alpha}, \mathbf{x}\rangle})\, e^{i\langle \boldsymbol{\lambda}-\boldsymbol{\mu}_w, \mathbf{x}\rangle} \mathrm{d}\mathbf{x} \\
= \; & \frac{1}{\mathrm{Vol}(\mathbb{T}_{\mathbf{R}})\,|W_{\boldsymbol{\mu}}|} \sum_{w\in W} \int_{\mathbb{T}_{\mathbf{R}}} e^{i\langle \boldsymbol{\lambda}-\boldsymbol{\mu}_w, \mathbf{x}\rangle} \prod_{\boldsymbol{\alpha}\in\mathbf{R}_+} (1-e^{i\langle \boldsymbol{\alpha}, \mathbf{x}\rangle}) \times \\
&\prod_{\boldsymbol{\alpha}\in\mathbf{R}^{(s)}_+} (1+\sum_{n=1}^{\infty} f^{(s)}_n e^{in\langle \boldsymbol{\alpha}, \mathbf{x}\rangle}) \prod_{\boldsymbol{\alpha}\in\mathbf{R}^{(l)}_+} (1+\sum_{n=1}^{\infty} f^{(l)}_n e^{in\langle \boldsymbol{\alpha}, \mathbf{x}\rangle}) \mathrm{d}\mathbf{x},
\end{aligned}$$

where $f^{(s)}_n$ and $f^{(l)}_n$ denote the coefficients in the Taylor series expansion of $1/c^{(s)}(z)$ and $1/c^{(l)}(z)$, respectively, around $z=0$. The integrals on the last two lines pick up the constant terms of the respective integrands multiplied by the volume of the torus $\mathbb{T}_{\mathbf{R}}$. A nonzero constant term can appear only when $\boldsymbol{\mu}_w \succeq \boldsymbol{\lambda}$ (for some $w \in W$). When $\boldsymbol{\mu} \not\succeq \boldsymbol{\lambda}$ one automatically has that $\boldsymbol{\mu}_w \not\succeq \boldsymbol{\lambda}$ for all $w \in W$ (since $\boldsymbol{\mu}_w \preceq \boldsymbol{\mu}$), whence the constant term vanishes in this case. On the other hand, when $\boldsymbol{\mu} = \boldsymbol{\lambda}$ the constant part of the term labeled by w is nonzero (namely equal to 1) if

and only if $w \in W_{\boldsymbol{\lambda}}$. By summing over all these contributions stemming from the stabilizer $W_{\boldsymbol{\lambda}}$ the proposition follows. □

Combination of Propositions 3.1 and 3.2 reveals that for $\boldsymbol{\lambda} \in \mathcal{P}_+$ sufficiently deep $P_{\boldsymbol{\lambda}}(\mathbf{x})$ (2.8a) coincides with the corresponding Bernstein-Szegö polynomial $p_{\boldsymbol{\lambda}}(\mathbf{x})$ defined by Eqs. (2.4a), (2.4b) up to normalization. Furthermore, since it is clear from Proposition 3.2 and the definition of the Bernstein-Szegö polynomials that $\langle P_{\boldsymbol{\lambda}}, p_{\boldsymbol{\mu}}\rangle_\Delta = 0$ for $\boldsymbol{\mu} \not\succeq \boldsymbol{\lambda} \in \mathcal{P}_+$, we conclude that $\langle p_{\boldsymbol{\lambda}}, p_{\boldsymbol{\mu}}\rangle_\Delta = 0$ for $\boldsymbol{\mu} \not\succeq \boldsymbol{\lambda} \in \mathcal{P}_+$ with $\boldsymbol{\lambda}$ sufficiently deep; the orthogonality stated in Theorem 2.2 then follows in view of the symmetry $\langle p_{\boldsymbol{\lambda}}, p_{\boldsymbol{\mu}}\rangle_\Delta = \overline{\langle p_{\boldsymbol{\mu}}, p_{\boldsymbol{\lambda}}\rangle_\Delta}$.

4. Normalization

To finish the proof of Theorem 2.1 it remains to verify that the leading coefficient of $P_{\boldsymbol{\lambda}}(\mathbf{x})$ (2.8a) is given by $\mathcal{N}_{\boldsymbol{\lambda}}$ (2.8b).

Proposition 4.1 (Leading Coefficient)**.** *The leading coefficient in the monomial expansion of $P_{\boldsymbol{\lambda}}(\mathbf{x})$ (2.8a) in Proposition 3.1 is given by $c_{\boldsymbol{\lambda}\boldsymbol{\lambda}} = \mathcal{N}_{\boldsymbol{\lambda}}$ (2.8b).*

Proof. The polynomial $P_{\boldsymbol{\lambda}}(\mathbf{x})$ (2.8a) reads explicitly

$$\frac{1}{\delta(\mathbf{x})}\sum_{w\in W}(-1)^w e^{i\langle\boldsymbol{\rho}+\boldsymbol{\lambda},\mathbf{x}_w\rangle}\prod_{\boldsymbol{\alpha}\in\mathbf{R}_+^{(s)}}\prod_{m=1}^{M^{(s)}}(1+t_m^{(s)}e^{-i\langle\boldsymbol{\alpha},\mathbf{x}_w\rangle})\prod_{\boldsymbol{\alpha}\in\mathbf{R}_+^{(l)}}\prod_{m=1}^{M^{(l)}}(1+t_m^{(l)}e^{-i\langle\boldsymbol{\alpha},\mathbf{x}_w\rangle}).$$

As was remarked in the proof of Proposition 3.1, this expression expands as a linear combination of terms of the form in Eq. (3.1), with $0 \le n_{\boldsymbol{\alpha}} \le M^{(s)} \le m^{(s)}(\boldsymbol{\rho}+\boldsymbol{\lambda})$ for $\boldsymbol{\alpha} \in \mathbf{R}_+^{(s)}$ and $0 \le n_{\boldsymbol{\alpha}} \le M^{(l)} \le m^{(l)}(\boldsymbol{\rho}+\boldsymbol{\lambda})$ for $\boldsymbol{\alpha} \in \mathbf{R}_+^{(l)}$. Upon replacing $\boldsymbol{\lambda}$ by $\boldsymbol{\lambda}+\boldsymbol{\rho}$ in Proposition 5.2 and Proposition 5.5 (below), it follows that in order for these terms to contribute to the leading monomial it is *necessary* that $n_{\boldsymbol{\alpha}} \in \{0, M^{(s)}\}$ for all $\boldsymbol{\alpha} \in \mathbf{R}_+^{(s)}$ and $n_{\boldsymbol{\alpha}} \in \{0, M^{(l)}\}$ for all $\boldsymbol{\alpha} \in \mathbf{R}_+^{(l)}$. From now on we will assume that both $M^{(s)}$ and $M^{(l)}$ are positive. (In the case that $M^{(s)}, M^{(l)} = 0$ one has that $P_{\boldsymbol{\lambda}}(\mathbf{x}) = \chi_{\boldsymbol{\lambda}}(\mathbf{x})$, whence $c_{\boldsymbol{\lambda}\boldsymbol{\lambda}} = 1$ trivially; the cases $M^{(s)} = 0$, $M^{(l)} \ge 1$ and $M^{(s)} \ge 1$, $M^{(l)} = 0$ can be recovered from the analysis below upon substituting formally $\mathbf{R}_+ = \mathbf{R}_+^{(l)}$ and $\mathbf{R}_+^{(s)} = \emptyset$ or $\mathbf{R}_+ = \mathbf{R}_+^{(s)}$ and $\mathbf{R}_+^{(l)} = \emptyset$, respectively.) The corresponding terms are then given explicitly by

$$\frac{1}{\delta(\mathbf{x})}\sum_{w\in W}(-1)^w\sum_{\mathbf{S}\subset\mathbf{R}_+}e^{i\langle\boldsymbol{\mu}(\mathbf{S}),\mathbf{x}_w\rangle}(-\mathbf{t}_s)^{\#(\mathbf{S}\cap\mathbf{R}_+^{(s)})}(-\mathbf{t}_l)^{\#(\mathbf{S}\cap\mathbf{R}_+^{(l)})},$$

with $\boldsymbol{\mu}(\mathbf{S}) := \boldsymbol{\rho}+\boldsymbol{\lambda}-M^{(s)}\sum_{\boldsymbol{\alpha}\in\mathbf{S}\cap\mathbf{R}_+^{(s)}}\boldsymbol{\alpha}-M^{(l)}\sum_{\boldsymbol{\alpha}\in\mathbf{S}\cap\mathbf{R}_+^{(l)}}\boldsymbol{\alpha}$ and $\mathbf{t}_s = -t_1^{(s)}\cdots t_{M^{(s)}}^{(s)}$, $\mathbf{t}_l = -t_1^{(l)}\cdots t_{M^{(l)}}^{(l)}$. Rewriting this expression in terms of Weyl characters $\chi_{\boldsymbol{\mu}}(\mathbf{x}) = \delta^{-1}(\mathbf{x})\sum_{w\in W}(-1)^w e^{\langle\boldsymbol{\rho}+\boldsymbol{\mu},\mathbf{x}_w\rangle}$, $\boldsymbol{\mu} \in \mathcal{P}_+$ produces

$$\sum_{\mathbf{S}\subset\mathbf{R}_+}(-1)^{w_{\mathbf{S}}}\chi_{\boldsymbol{\lambda}(\mathbf{S})}(\mathbf{x})(-\mathbf{t}_s)^{\#(\mathbf{S}\cap\mathbf{R}_+^{(s)})}(-\mathbf{t}_l)^{\#(\mathbf{S}\cap\mathbf{R}_+^{(l)})},$$

where $w_{\mathbf{S}}$ denotes the unique shortest Weyl group element permuting $\boldsymbol{\mu}(\mathbf{S})$ into the dominant cone $\mathcal{P}_+$ and $\boldsymbol{\lambda}(\mathbf{S}) := w_{\mathbf{S}}(\boldsymbol{\mu}(\mathbf{S})) - \boldsymbol{\rho}$ (here we have also assumed the convention that the Weyl character $\chi_{\boldsymbol{\lambda}(\mathbf{S})}(\mathbf{x})$ vanishes when $\boldsymbol{\lambda}(\mathbf{S})$ is not dominant). The contributions to the leading monomial stem from those subsets $\mathbf{S} \subset \mathbf{R}_+$ for which $\boldsymbol{\lambda}(\mathbf{S}) = \boldsymbol{\lambda}$, or equivalently, $\boldsymbol{\mu}(\mathbf{S}) \in W(\boldsymbol{\rho}+\boldsymbol{\lambda})$. From Proposition 5.7 (below)

with $\boldsymbol{\lambda}$ replaced by $\boldsymbol{\lambda}+\boldsymbol{\rho}$, it follows that these are precisely those subsets $\mathbf{S}\subset\mathbf{R}_+$ of the form $\mathbf{S}=\mathbf{S}_w:=\{\boldsymbol{\alpha}\in\mathbf{R}_+\mid w(\boldsymbol{\alpha})\not\in\mathbf{R}_+\}$ for some $w\in W_{\tilde{\boldsymbol{\lambda}}}$, where $\tilde{\boldsymbol{\lambda}}:=\boldsymbol{\rho}+\boldsymbol{\lambda}-M^{(s)}\boldsymbol{\rho}^{(s)}-M^{(l)}\boldsymbol{\rho}^{(l)}$ (cf. in this connection also the remark just after Proposition 5.7). By summing over all contributions from the subsets $\mathbf{S}_w$, $w\in W_{\tilde{\boldsymbol{\lambda}}}$ (and recalling the fact that the monomial expansion of the Weyl character $\chi_{\boldsymbol{\lambda}}(\mathbf{x})$ is monic with leading term $m_{\boldsymbol{\lambda}}(\mathbf{x})$), one concludes that the leading coefficient $c_{\boldsymbol{\lambda}\boldsymbol{\lambda}}$ in the monomial expansion of $P_{\boldsymbol{\lambda}}(\mathbf{x})$ is given by the following Poincaré type series of the stabilizer $W_{\tilde{\boldsymbol{\lambda}}}$:

$$c_{\boldsymbol{\lambda}\boldsymbol{\lambda}}=\sum_{w\in W_{\tilde{\boldsymbol{\lambda}}}}\mathbf{t}_s^{\ell_s(w)}\mathbf{t}_l^{\ell_l(w)},$$

where $\ell_s(w):=\#\{\boldsymbol{\alpha}\in\mathbf{R}_+^{(s)}\mid w(\boldsymbol{\alpha})\not\in\mathbf{R}_+^{(s)}\}$, $\ell_l(w):=\#\{\boldsymbol{\alpha}\in\mathbf{R}_+^{(l)}\mid w(\boldsymbol{\alpha})\not\in\mathbf{R}_+^{(l)}\}$. (Notice in this respect that the minus signs dropped out as $(-1)^{w\mathbf{s}}=(-1)^{\ell_s(w\mathbf{s})+\ell_l(w\mathbf{s})}=(-1)^{\#S}$.) Invoking a general product formula for the (two-parameter) Poincaré series of Weyl groups due to Macdonald [M2, Theorem (2.4)] then gives rise to

$$c_{\boldsymbol{\lambda}\boldsymbol{\lambda}}=\prod_{\substack{\boldsymbol{\alpha}\in\mathbf{R}_+^{(s)}\\ \langle\tilde{\boldsymbol{\lambda}},\boldsymbol{\alpha}^\vee\rangle=0}}\frac{1-\mathbf{t}_s^{1+\mathrm{ht}_s(\boldsymbol{\alpha})}\mathbf{t}_l^{\mathrm{ht}_l(\boldsymbol{\alpha})}}{1-\mathbf{t}_s^{\mathrm{ht}_s(\boldsymbol{\alpha})}\mathbf{t}_l^{\mathrm{ht}_l(\boldsymbol{\alpha})}}\prod_{\substack{\boldsymbol{\alpha}\in\mathbf{R}_+^{(l)}\\ \langle\tilde{\boldsymbol{\lambda}},\boldsymbol{\alpha}^\vee\rangle=0}}\frac{1-\mathbf{t}_s^{\mathrm{ht}_s(\boldsymbol{\alpha})}\mathbf{t}_l^{1+\mathrm{ht}_l(\boldsymbol{\alpha})}}{1-\mathbf{t}_s^{\mathrm{ht}_s(\boldsymbol{\alpha})}\mathbf{t}_l^{\mathrm{ht}_l(\boldsymbol{\alpha})}},$$

where $\mathrm{ht}_s(\boldsymbol{\alpha})=\sum_{\boldsymbol{\beta}\in\mathbf{R}_+^{(s)}}\langle\boldsymbol{\alpha},\boldsymbol{\beta}^\vee\rangle/2$ and $\mathrm{ht}_l(\boldsymbol{\alpha})=\sum_{\boldsymbol{\beta}\in\mathbf{R}_+^{(l)}}\langle\boldsymbol{\alpha},\boldsymbol{\beta}^\vee\rangle/2$, which completes the proof of the proposition. □

Finally, by combining Propositions 3.1, 3.2, and 4.1, the norm formula in Theorem 2.3 readily follows: $\langle p_{\boldsymbol{\lambda}},p_{\boldsymbol{\lambda}}\rangle_\Delta=\mathcal{N}_{\boldsymbol{\lambda}}^{-2}\langle P_{\boldsymbol{\lambda}},P_{\boldsymbol{\lambda}}\rangle_\Delta=\mathcal{N}_{\boldsymbol{\lambda}}^{-1}\langle P_{\boldsymbol{\lambda}},m_{\boldsymbol{\lambda}}\rangle_\Delta=\mathcal{N}_{\boldsymbol{\lambda}}^{-1}$ (for $\boldsymbol{\lambda}$ sufficiently deep in the dominant cone).

5. Saturated sets of weights

In the proof of Propositions 3.1 and 4.1 we exploited geometric properties of saturated subsets of the weight lattice that are of interest in their own right independent of the current application. To formulate these properties some additional notation is required. To a dominant weight $\boldsymbol{\lambda}$, we associated the following finite subsets of the weight lattice

$$\mathcal{P}_+(\boldsymbol{\lambda}):=\{\boldsymbol{\mu}\in\mathcal{P}_+\mid\boldsymbol{\mu}\preceq\boldsymbol{\lambda}\},\qquad\mathcal{P}(\boldsymbol{\lambda}):=\bigcup_{\boldsymbol{\mu}\in\mathcal{P}_+(\boldsymbol{\lambda})}W(\boldsymbol{\mu}).\tag{5.1}$$

The subset $\mathcal{P}(\boldsymbol{\lambda})$ is *saturated*, i.e. for each $\boldsymbol{\mu}\in\mathcal{P}(\boldsymbol{\lambda})$ and $\boldsymbol{\alpha}\in\mathbf{R}$ the $\boldsymbol{\alpha}$-string through $\boldsymbol{\mu}$ of the form $\{\boldsymbol{\mu}-\ell\boldsymbol{\alpha}\mid\ell=0,\ldots,\langle\boldsymbol{\mu},\boldsymbol{\alpha}^\vee\rangle\}$ belongs to $\mathcal{P}(\boldsymbol{\lambda})$ [B, Hu]. Reversely, any saturated subset of the weight lattice containing $\boldsymbol{\lambda}$ necessarily contains the whole of $\mathcal{P}(\boldsymbol{\lambda})$ (5.1).

It is known from the representation theory of simple Lie algebras that $\mathcal{P}(\boldsymbol{\lambda})$ (5.1) lies inside the convex hull of the Weyl-orbit through the highest weight vector $\boldsymbol{\lambda}$. More precisely, we have the following geometric characterization of $\mathcal{P}(\boldsymbol{\lambda})$ taken from Ref. [K, Prop. 11.3, part a)].

Lemma 5.1 ([K])**.** *The saturated set $\mathcal{P}(\boldsymbol{\lambda})$ (5.1) amounts to the points of the translated root lattice $\boldsymbol{\lambda}+\mathcal{Q}$ inside the convex hull of the Weyl-orbit $W(\boldsymbol{\lambda})$:*

$$\mathcal{P}(\boldsymbol{\lambda})=\mathit{Conv}(W(\boldsymbol{\lambda}))\cap(\boldsymbol{\lambda}+\mathcal{Q}).$$

Since any dominant weight is maximal in its Weyl orbit (see e.g. Ref. [Hu, Sec. 13.2]), it is clear from this lemma that all weights of $\mathcal{P}(\boldsymbol{\lambda})$ (5.1) are obtained from $\boldsymbol{\lambda}$ via iterated subtraction of positive roots. The following proposition provides quantitative information on the number of times positive roots may be subtracted from $\boldsymbol{\lambda}$ without leaving the convex hull of $W(\boldsymbol{\lambda})$.

Proposition 5.2. *Let $\boldsymbol{\lambda} \in \mathcal{P}_+$ and let $n_{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} \in \mathbf{R}_+$ be integers such that $0 \leq n_{\boldsymbol{\alpha}} \leq m^{(s)}(\boldsymbol{\lambda})$, $\forall \boldsymbol{\alpha} \in \mathbf{R}_+^{(s)}$ and $0 \leq n_{\boldsymbol{\alpha}} \leq m^{(l)}(\boldsymbol{\lambda})$, $\forall \boldsymbol{\alpha} \in \mathbf{R}_+^{(l)}$. Then one has that*

$$\boldsymbol{\lambda} - \sum_{\boldsymbol{\alpha} \in \mathbf{R}_+} n_{\boldsymbol{\alpha}} \boldsymbol{\alpha} \in \mathcal{P}(\boldsymbol{\lambda}).$$

The proof of this proposition hinges on two lemmas.

Lemma 5.3. *For any $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{P}_+$ the following inclusion holds*

$$\boldsymbol{\mu} + \mathit{Conv}(W(\boldsymbol{\nu})) \subset \mathit{Conv}(W(\boldsymbol{\mu} + \boldsymbol{\nu})).$$

Proof. Clearly it suffices to show that $\boldsymbol{\mu} + W(\boldsymbol{\nu}) \subset \mathrm{Conv}(W(\boldsymbol{\mu} + \boldsymbol{\nu}))$. Since all weights in $W(\boldsymbol{\mu}) + W(\boldsymbol{\nu})$ are smaller than or equal to $\boldsymbol{\mu} + \boldsymbol{\nu}$, it is evident that the intersection of $W(\boldsymbol{\mu}) + W(\boldsymbol{\nu})$ with the cone of dominant weights $\mathcal{P}_+$ is contained in $\mathcal{P}_+(\boldsymbol{\mu}+\boldsymbol{\nu})$. We thus conclude that $\boldsymbol{\mu} + W(\boldsymbol{\nu}) \subset W(\boldsymbol{\mu}) + W(\boldsymbol{\nu}) \subset \mathcal{P}(\boldsymbol{\mu}+\boldsymbol{\nu})$. But then we have in particular that $\boldsymbol{\mu} + W(\boldsymbol{\nu}) \subset \mathrm{Conv}(W(\boldsymbol{\mu} + \boldsymbol{\nu}))$ in view of Lemma 5.1., whence the inclusion stated in the lemma follows. □

Lemma 5.4. *Let $a, b \geq 0$ and let $\boldsymbol{\rho}^{(s)}$, $\boldsymbol{\rho}^{(l)}$ be given by Eq. (2.6). Then the convex hull of $W(a\boldsymbol{\rho}^{(s)} + b\boldsymbol{\rho}^{(l)})$ reads explicitly*

$$\begin{aligned} &\mathit{Conv}(W(a\boldsymbol{\rho}^{(s)} + b\boldsymbol{\rho}^{(l)})) = \\ &\qquad \Big\{ a \sum_{\boldsymbol{\alpha} \in \mathbf{R}_+^{(s)}} t_{\boldsymbol{\alpha}} \boldsymbol{\alpha} + b \sum_{\boldsymbol{\alpha} \in \mathbf{R}_+^{(l)}} t_{\boldsymbol{\alpha}} \boldsymbol{\alpha} \mid -\tfrac{1}{2} \leq t_{\boldsymbol{\alpha}} \leq \tfrac{1}{2}, \boldsymbol{\alpha} \in \mathbf{R}_+ \Big\}. \end{aligned}$$

Proof. The r.h.s. is manifestly convex, Weyl-group invariant, and it contains the vertex $a\boldsymbol{\rho}^{(s)} + b\boldsymbol{\rho}^{(l)}$. We thus conclude that the l.h.s. is a subset of the r.h.s. Furthermore, the intersection of the l.h.s. with the closure of the dominant Weyl chamber $\mathbf{C} := \{\mathbf{x} \in \mathbf{E} \mid \langle \mathbf{x}, \boldsymbol{\alpha} \rangle \geq 0,\ \forall \boldsymbol{\alpha} \in \mathbf{R}_+\}$ consists of all vectors in $\mathbf{C}$ that can be obtained from the vertex $a\boldsymbol{\rho}^{(s)} + b\boldsymbol{\rho}^{(l)}$ by subtracting nonnegative linear combinations of the positive roots. (This is because the image of the vertex $a\boldsymbol{\rho}^{(s)} + b\boldsymbol{\rho}^{(l)}$ with respect to the orthogonal reflection in a wall of the dominant chamber is obtained by subtracting a nonnegative multiple of the corresponding simple root perpendicular to the wall in question.) Hence, the intersection of $\mathbf{C}$ with the r.h.s. is contained in the intersection of $\mathbf{C}$ with the l.h.s. But then the r.h.s. must be a subset of the l.h.s. as both sides are Weyl-group invariant (and the closure of the dominant Weyl chamber $\mathbf{C}$ constitutes a fundamental domain for the action of the Weyl group on $\mathbf{E}$). □

To prove Proposition 5.2, we apply Lemma 5.3 with

$$\boldsymbol{\mu} = \boldsymbol{\lambda} - m^{(s)}\boldsymbol{\rho}^{(s)} - m^{(l)}\boldsymbol{\rho}^{(l)} \quad \text{and} \quad \boldsymbol{\nu} = m^{(s)}\boldsymbol{\rho}^{(s)} + m^{(l)}\boldsymbol{\rho}^{(l)}, \tag{5.2}$$

where $m^{(s)} = m^{(s)}(\boldsymbol{\lambda})$ and $m^{(l)} = m^{(l)}(\boldsymbol{\lambda})$, respectively. Upon computing $\mathrm{Conv}(W(\boldsymbol{\nu}))$ with the aid of Lemma 5.4 this entails the inclusion

$$\begin{aligned}\boldsymbol{\lambda} - \Bigl\{ m^{(s)} \sum_{\boldsymbol{\alpha}\in\mathbf{R}_+^{(s)}} t_{\boldsymbol{\alpha}}\boldsymbol{\alpha} + m^{(l)} \sum_{\boldsymbol{\alpha}\in\mathbf{R}_+^{(l)}} t_{\boldsymbol{\alpha}}\boldsymbol{\alpha} \mid 0 \leq t_{\boldsymbol{\alpha}} \leq 1, \boldsymbol{\alpha} \in \mathbf{R}_+ \Bigr\} \\ \subset \mathrm{Conv}(W(\boldsymbol{\lambda})),\end{aligned} \tag{5.3}$$

which implies Proposition 5.2 in view of Lemma 5.1.

The vertices of the convex set on the r.h.s. of Eq. (5.3) are given by the orbit $W(\boldsymbol{\lambda})$, whereas for a point to be a vertex of the convex set on the l.h.s. it is necessary that $t_{\boldsymbol{\alpha}} \in \{0, 1\}$, $\forall \boldsymbol{\alpha} \in \mathbf{R}_+$. This observation gives rise to the following additional information regarding the weights in Proposition 5.2 lying on the highest-weight orbit $W(\boldsymbol{\lambda})$.

Proposition 5.5. *Let $\boldsymbol{\lambda} \in \mathcal{P}_+$ and let $n_{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} \in \mathbf{R}_+$ be integers such that $0 \leq n_{\boldsymbol{\alpha}} \leq m^{(s)}(\boldsymbol{\lambda})$, $\forall \boldsymbol{\alpha} \in \mathbf{R}_+^{(s)}$ and $0 \leq n_{\boldsymbol{\alpha}} \leq m^{(l)}(\boldsymbol{\lambda})$, $\forall \boldsymbol{\alpha} \in \mathbf{R}_+^{(l)}$. Then*

$$\boldsymbol{\lambda} - \sum_{\boldsymbol{\alpha}\in\mathbf{R}_+} n_{\boldsymbol{\alpha}}\boldsymbol{\alpha} \in W(\boldsymbol{\lambda})$$

implies that $n_{\boldsymbol{\alpha}} \in \{0, m^{(s)}(\boldsymbol{\lambda})\}$, $\forall \boldsymbol{\alpha} \in \mathbf{R}_+^{(s)}$ and $n_{\boldsymbol{\alpha}} \in \{0, m^{(l)}(\boldsymbol{\lambda})\}$, $\forall \boldsymbol{\alpha} \in \mathbf{R}_+^{(l)}$.

Much weaker versions of the statements in Proposition 5.2 and Proposition 5.5 can be found in the appendix of Ref. [Di]. For the root systems of type A Proposition 5.2 and a somewhat weaker form of Proposition 5.5 were verified in Ref. [R] by means of an explicit combinatorial analysis.

Proposition 5.5 provides a necessary condition on the coefficients $n_{\boldsymbol{\alpha}}$ such that a weight in Proposition 5.2 lies on the highest-weight orbit $W(\boldsymbol{\lambda})$. We will now wrap up with a more precise characterization of the weights in question when $\boldsymbol{\lambda}$ is strongly dominant.

Lemma 5.6. *For any $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{P}_+$ with $\boldsymbol{\nu}$ strongly dominant (i.e. with $m^{(s)}(\boldsymbol{\nu})$, $m^{(l)}(\boldsymbol{\nu})$ strictly positive), the intersection of $\boldsymbol{\mu} + W(\boldsymbol{\nu})$ and $W(\boldsymbol{\mu} + \boldsymbol{\nu})$ is given by*

$$(\boldsymbol{\mu} + W(\boldsymbol{\nu})) \cap W(\boldsymbol{\mu} + \boldsymbol{\nu}) = \boldsymbol{\mu} + W_{\boldsymbol{\mu}}(\boldsymbol{\nu}).$$

Proof. The r.h.s. is manifestly contained in the intersection on the l.h.s. It is therefore sufficient to demonstrate that the l.h.s. is also a subset of the r.h.s. The intersection on the l.h.s. consists of those weights such that $\boldsymbol{\mu} + w_1(\boldsymbol{\nu}) = w_2(\boldsymbol{\mu} + \boldsymbol{\nu})$ for some $w_1, w_2 \in W$. This implies that $\|\boldsymbol{\mu} + w_1(\boldsymbol{\nu})\|^2 = \|\boldsymbol{\mu} + \boldsymbol{\nu}\|^2$, or equivalently, $\langle \boldsymbol{\mu} - w_1^{-1}(\boldsymbol{\mu}), \boldsymbol{\nu} \rangle = 0$. But then we must have that $w_1(\boldsymbol{\mu}) = \boldsymbol{\mu}$ (and thus $w_2 = w_1$) since $\boldsymbol{\mu} - w_1^{-1}(\boldsymbol{\mu}) \in \mathcal{Q}_+$ and $\boldsymbol{\nu}$ is strongly dominant. It thus follows that the weights in question form part of the r.h.s. □

By specializing Lemma 5.6 to weights $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ of the form in Eq. (5.2) with $\boldsymbol{\lambda}$ strongly dominant, and with $0 < m^{(s)} \leq m^{(s)}(\boldsymbol{\lambda})$ and $0 < m^{(l)} \leq m^{(l)}(\boldsymbol{\lambda})$, we arrive at the following proposition.

Proposition 5.7. *Let $\boldsymbol{\lambda} \in \mathcal{P}_+$ be strongly dominant and let $n_{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} \in \mathbf{R}_+$ be integers such that $0 \leq n_{\boldsymbol{\alpha}} \leq m^{(s)}$, $\forall \boldsymbol{\alpha} \in \mathbf{R}_+^{(s)}$ and $0 \leq n_{\boldsymbol{\alpha}} \leq m^{(l)}$, $\forall \boldsymbol{\alpha} \in \mathbf{R}_+^{(l)}$, where $0 < m^{(s)} \leq m^{(s)}(\boldsymbol{\lambda})$ and $0 < m^{(l)} \leq m^{(l)}(\boldsymbol{\lambda})$. Then $\boldsymbol{\lambda} - \sum_{\boldsymbol{\alpha}\in\mathbf{R}_+} n_{\boldsymbol{\alpha}}\boldsymbol{\alpha} \in W(\boldsymbol{\lambda})$ if*

and only if it is of the form

$$\boldsymbol{\lambda} - m^{(s)} \sum_{\boldsymbol{\alpha}\in \mathbf{S}_w\cap \mathbf{R}_+^{(s)}} \boldsymbol{\alpha} - m^{(l)} \sum_{\boldsymbol{\alpha}\in \mathbf{S}_w\cap \mathbf{R}_+^{(l)}} \boldsymbol{\alpha},$$

where $\mathbf{S}_w := \{\boldsymbol{\alpha}\in\mathbf{R}_+ \mid w(\boldsymbol{\alpha})\not\in\mathbf{R}_+\}$ *for some* $w\in W_{\tilde{\boldsymbol{\lambda}}}$ *with*

$$\tilde{\boldsymbol{\lambda}} := \boldsymbol{\lambda} - m^{(s)}\boldsymbol{\rho}^{(s)} - m^{(l)}\boldsymbol{\rho}^{(l)}.$$

Proof. The weights characterized by the premises of the proposition consist of the common vertices of the convex sets on both sides of Eq. (5.3). It is immediate from the previous discussion that the vertices at issue are given by the weights in the intersection of $\boldsymbol{\mu}+W(\boldsymbol{\nu})$ and $W(\boldsymbol{\mu}+\boldsymbol{\nu})$, with $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ given by Eq. (5.2). According to Lemma 5.6, this intersection consists of all weights of the form

$$\begin{aligned}&\boldsymbol{\lambda} - m^{(s)}\boldsymbol{\rho}^{(s)} - m^{(l)}\boldsymbol{\rho}^{(l)} + w^{-1}(m^{(s)}\boldsymbol{\rho}^{(s)} + m^{(l)}\boldsymbol{\rho}^{(l)})\\ &\quad = \boldsymbol{\lambda} - m^{(s)} \sum_{\substack{\boldsymbol{\alpha}\in \mathbf{R}_+^{(s)}\\ w(\boldsymbol{\alpha})\not\in\mathbf{R}_+^{(s)}}} \boldsymbol{\alpha} - m^{(l)} \sum_{\substack{\boldsymbol{\alpha}\in \mathbf{R}_+^{(l)}\\ w(\boldsymbol{\alpha})\not\in\mathbf{R}_+^{(l)}}} \boldsymbol{\alpha},\end{aligned}$$

where w runs through the stabilizer of the weight $\boldsymbol{\lambda} - m^{(s)}\boldsymbol{\rho}^{(s)} - m^{(l)}\boldsymbol{\rho}^{(l)}$. □

Remark. Proposition 5.7 implies Proposition 5.5 for $\boldsymbol{\lambda}$ strongly dominant. Indeed, the stabilizer $W_{\tilde{\boldsymbol{\lambda}}}$ is generated by the reflections in the short simple roots $\boldsymbol{\alpha}$ such that $\langle\boldsymbol{\lambda},\boldsymbol{\alpha}^\vee\rangle = m^{(s)}$ (these reflections permute the roots of $\mathbf{R}_+^{(l)}$) and by the reflections in the long simple roots $\boldsymbol{\alpha}$ such that $\langle\boldsymbol{\lambda},\boldsymbol{\alpha}^\vee\rangle = m^{(l)}$ (these reflections permute the roots of $\mathbf{R}_+^{(s)}$). Hence, for $m^{(s)} < m^{(s)}(\boldsymbol{\lambda})$ or $m^{(l)} < m^{(l)}(\boldsymbol{\lambda})$ one has that $\mathbf{S}_w\cap\mathbf{R}_+^{(s)} = \emptyset$ or $\mathbf{S}_w\cap\mathbf{R}_+^{(l)} = \emptyset$, respectively. In other words, nonvanishing contributions to the sums in the formula of Proposition 5.7 only arise where $m^{(s)}$ or $m^{(l)}$ assume their maximum values $m^{(s)}(\boldsymbol{\lambda})$ and $m^{(l)}(\boldsymbol{\lambda})$, respectively.

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